

ON THE DETERMINATION OF THE PHASE OF A FOURIER INTEGRAL, I⁽¹⁾

BY

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1. **Introduction.** Suppose $\phi(t)$ is a complex valued function on $(-\infty, \infty)$ and let $\hat{\phi}(x)$ denote its Fourier transform. A question which arises in various physical applications, and which has an intrinsic interest in its own right is: To what extent does the modulus of $\hat{\phi}$ determine ϕ ? It is inevitable that some a priori conditions be imposed in order to obtain reasonably determinate results, and we shall establish the following

THEOREM 1. *Let $\mathcal{C}(a)$ be the class of all functions ϕ fulfilling the following conditions:*

α . $\phi \in L^1 \cap L^2$, where L^1 and L^2 are the usual Lebesgue function spaces on $(-\infty, \infty)$,

β . $\phi(t)$ vanishes almost everywhere for $t < 0$ ⁽²⁾,

γ . $\hat{\phi}(x) \neq 0$, $-\infty < x < \infty$,

δ . $a(x)$ is a fixed function such that $|\hat{\phi}(x)| = a(x)$, $-\infty < x < \infty$. Then if ϕ_1 and ϕ_2 belong to $\mathcal{C}(a)$ there subsists a relation between them of the form

$$(1) \quad e^{ic_1 + ib_1x} B_1(x) \hat{\phi}_1(x) = e^{ic_2 + ib_2x} B_2(x) \hat{\phi}_2(x),$$

where c_1, c_2, b_1, b_2 , are real numbers, $b_1 \geq 0, b_2 \geq 0$ and $B_1(x), B_2(x)$ are limits as $y \rightarrow 0+$ of certain Blaschke products in the upper half-plane. $B_1(x)$ and $B_2(x)$ are holomorphic functions of modulus identically 1.

Thus ϕ and $\hat{\phi}$ are both partly specified, and the data fall short of determining ϕ (or $\hat{\phi}$) uniquely to the extent of the arbitrariness of the zeros occurring in the Blaschke products, a complex number of modulus 1, and a pure oscillation. The zeros of B_1 and B_2 are not entirely arbitrary (subject, of course, to the convergence of the products), for the regularity of B_1 and B_2 on the real axis excludes the existence of a cluster point of zeros at a real point.

The method of proof depends in an essential way upon a canonical representation of certain holomorphic functions of one variable in the upper half-plane, and the multi-dimensional case remains untouched. This is perhaps

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⁽²⁾ $t < 0$ can be replaced by $t < t_0(\phi)$, for a translation reduces the latter case to the former; the conclusion is altered only by the addition of $t_0(\phi_j)$ to b_j , $j = 1, 2$.

unfortunate from the standpoint of the crystallographers. (See §6, Example II.)

It would be possibly interesting to determine what happens if $\hat{\phi}$ is allowed to vanish.

2. Statement of several known facts. We begin by recalling certain facts about Blaschke products in the upper half-plane. Suppose $a_1, a_2, \dots$ is a sequence of complex numbers with $\text{Im } a_k > 0$, and

$$(2) \quad \sum_{k=1}^{\infty} \frac{\text{Im } a_k}{1 + |a_k|^2} < \infty.$$

Condition (2) is necessary and sufficient that the Blaschke product,

$$\left(\frac{z-i}{z+i} \right)^n \prod_{k=1}^{\infty} \frac{|a_k-i|}{a_k-i} \cdot \frac{|a_k+i|}{a_k+i} \cdot \frac{z-a_k}{z-\bar{a}_k},$$

(where $z = x + iy$ and $n = \text{non-negative integer}$) be convergent for $y > 0$ to a holomorphic function $B(z)$. Then $|B(z)| < 1$ and $\lim_{y \rightarrow 0} |B(x + iy)| = 1$ for almost all x . If there are no zeros we set $B(z) \equiv 1$. Such a function has a simple characterization, embodied in the following theorem.

(A) [1] Suppose 1. $F(z)$ is holomorphic for $y > 0$, $|F(z)| \leq 1$, and 2.

$$\lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \frac{\log |F(x + iy)|}{1 + x^2} dx = 0.$$

Then $F(z)$ is of the form

$$F(z) = e^{ic + i\beta z} B(z),$$

where c and β are real, $\beta \geq 0$, and $B(z)$ is the Blaschke product formed with the zeros of F . Conversely, if $B(z)$ is any Blaschke product in the upper half-plane, 2. holds with F replaced by B .

Subsequent arguments rest upon a number of more or less well-known theorems and formulae, which we proceed to assemble. These need not be read before they are referred to later on.

(B) [2, pp. 18–20]. If ϕ belongs to $L^2(-\infty, \infty)$ and vanishes on a half-line, then

$$\int_{-\infty}^{\infty} \frac{|\log |\hat{\phi}(x)||}{1 + x^2} dx < \infty.$$

(C) [3, Theorem 1, p. 643]. If $\phi(t)/(1+t^2)$ belongs to $L^1(-\infty, \infty)$ and is continuous at x_0 , then

$$u(x, y) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} \phi(t) dt$$

is harmonic for $y > 0$ and $\lim_{y \rightarrow 0} u(x_0, y) = \phi(x_0)$.

An examination of the proof of (C) will disclose that the limit is uniform with respect to x in closed intervals interior to intervals of continuity of ϕ .

(D) [4, p. 106, Theorem IX for $p=2$]. Let $\Phi(z)$ be a holomorphic function for $y > 0$ subject to

$$(3) \quad \int_{-\infty}^{\infty} |\Phi(x + iy)|^2 dx \leq M < \infty,$$

where M is independent of y . $\Phi(z)$ can then be represented in the form

$$(4) \quad \Phi(z) = e^{ic + i\beta z} B(z) D(z) G(z),$$

where

- (i) c is a real number,
- (ii) β is a non-negative real number,
- (iii) $B(z)$ is the (convergent) Blaschke product formed with the zeros of Φ ,

$$(iv) \quad D(z) = \exp \left(\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \frac{\log |\Phi(t)|}{1 + t^2} dt \right),$$

where $\Phi(x) = \lim_{y \rightarrow 0} \Phi(x + iy)$ almost everywhere, and

$$(v) \quad \int_{-\infty}^{\infty} \frac{|\log |\Phi(x)||}{1 + x^2} dx < \infty, \quad \int_{-\infty}^{\infty} |\Phi(x)|^2 dx < \infty,$$

$$G(z) = \exp \left(\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} dE(t) \right),$$

where $E(t)$ is a real, bounded, increasing function with derivative $E'(t) = 0$ almost everywhere. Conversely, every function of the form (4) is holomorphic for $y > 0$ and satisfies (3).

(E) [5, p. 44]. If $\Phi(z)$ is holomorphic for $y > 0$ and in the neighborhood of every point on the real axis satisfies $\limsup |\Phi(z)| \leq 1$, then either

(α) the modulus $|\Phi(z)|$ tends to $+\infty$ so rapidly that

$$\liminf_{r \rightarrow \infty} \frac{\log M(r)}{r} > 0,$$

where $M(r) = \max_{|z|=r} |\Phi(z)|$, or

(β) $|\Phi(z)| \leq 1$ for $y > 0$.

(F) [6, p. 152]. If both the upper and lower symmetrical derivatives of a function of bounded variation are everywhere finite, then the function is absolutely continuous.

3. Proof of the Theorem 1. Let ϕ denote any function of the class $\mathcal{C}(a)$. Define a holomorphic function of $z = x + iy$ for $y > 0$ by

$$\Phi(z) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i(x+iy)t} \phi(t) dt.$$

We shall refer to Φ as the *holomorphic extension of $\hat{\phi}$* . Then by the Parseval identity and Schwarz inequality:

$$\begin{aligned} \int_{-\infty}^{\infty} |\Phi(x+iy)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_0^\infty e^{ixt-y|t|} \phi(t) dt \right|^2 dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \left(\frac{2}{\pi}\right)^{1/2} \frac{y}{y^2 + (x+\lambda)^2} \widehat{\phi}(\lambda) d\lambda \right|^2 dx \\ &\leq \frac{1}{\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{y}{y^2 + (x+\lambda)^2} d\lambda \int_{-\infty}^{\infty} \frac{y}{y^2 + (x+\lambda)^2} |a(\lambda)|^2 d\lambda \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{y}{y^2 + (x+\lambda)^2} |a(\lambda)|^2 d\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} |a(\lambda)|^2 d\lambda < \infty. \end{aligned}$$

By (D),

$$(4) \quad \Phi(z) = e^{ic+iyz} B(z) D(z) G(z),$$

where c, β, B, D, G have the properties mentioned in §2. The next paragraph is devoted to showing that $G(z) \equiv 1$; that is, $E(t) \equiv \text{const.}$

In the first place,

$$|D(z)| = \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (t-x)^2} \log a(t) dt \right),$$

because $\Phi(x+iy) \rightarrow \hat{\phi}(x)$ as $y \rightarrow 0$, for every x . By (B) and (C) it follows that

$$|D(z)| \rightarrow a(x) \quad \text{as } y \rightarrow 0, \quad -\infty < x < \infty.$$

Since we have $|B(z)| < 1$, (4) implies:

$$\begin{aligned} a(x) = \lim | \Phi(x+iy) | &\leq \liminf \left| \frac{\Phi(x+iy)}{B(x+iy)} \right| \\ &= a(x) \liminf \exp \left(- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{y(1+t^2)}{y^2 + (x-t)^2} dE(t) \right). \end{aligned}$$

Therefore, as $a(x) \neq 0$,

$$\limsup \int_{-\infty}^{\infty} \frac{y(1+t^2)}{y^2 + (x-t)^2} dE(t) < \infty, \quad -\infty < x < \infty.$$

A fortiori,

$$(5) \quad \limsup \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} dE(t) < \infty, \quad -\infty < x < \infty.$$

Since $E(x+t) - E(x-t)$ is an increasing function of t , we can write:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-t)^2} dE(t) &= \int_0^{\infty} \frac{y}{y^2 + t^2} d[E(x+t) - E(x-t)] \\ &\geq \int_0^y \frac{y}{y^2 + t^2} d[E(x+t) - E(x-t)] \\ &\geq \frac{E(x+y) - E(x-y)}{2y}, \quad -\infty < x < \infty. \end{aligned}$$

Therefore in view of (5), the upper (and lower) symmetrical derivatives of E are everywhere finite. By (F), E is absolutely continuous. Since $E' = 0$ almost everywhere, $E \equiv \text{const.}$, $G \equiv 1$.

Hence, if ϕ belongs to the class $\mathcal{C}(a)$, we have

$$(6) \quad \Phi(z) = e^{ic + i\theta z} B(z) D(z).$$

From (6), together with the condition $\hat{\phi}(x) \neq 0$, $-\infty < x < \infty$, it follows that $B(z)$ cannot have zeros which cluster at a finite point of the real axis. But then $B(z)$ is uniformly convergent in a rectangle $|y| \leq \delta$, $a \leq x \leq b$, provided δ is so small that this rectangle is free of zeros^(*). Denoting the zeros of $B(z)$ by $a_k = x_k + iy_k$, this follows at once from the identity

$$\frac{|a_k + i|}{a_k + i} \cdot \frac{|a_k - i|}{a_k - i} \cdot \frac{z - a_k}{z - \bar{a}_k} = 1 - \frac{iy_k}{1 + x_k^2 + y_k^2} \cdot \frac{(\bar{i} + a_k)(i + z)}{z - \bar{a}_k} \cdot c_k,$$

where

$$c_k = \frac{1 + |(i - a_k)/(i + a_k)|^2}{1 + |(i - a_k)/(i + a_k)|} \cdot \left\{ 1 + \frac{i - z}{i + z} \cdot \frac{i + a_k}{i - a_k} \cdot \left| \frac{i - a_k}{i + a_k} \right| \right\}.$$

Therefore $B(z)$ is holomorphic for $y \geq 0$. By (6),

$$\lim_{y \rightarrow 0} D(z) \equiv D(x), \quad -\infty < x < \infty,$$

exists as a continuous function, and the conclusion of Theorem 1 follows at once from (6), written for ϕ_1 and ϕ_2 with $y = 0$.

REMARK. It can be shown that

$$D(x) = a(x) \exp iH(x),$$

where

(*) I owe this remark to a referee.

$$H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1+tx}{t-x} \cdot \frac{\log a(t)}{1+t^2} dt,$$

the integral being taken as a principal value at $t=x$. We omit the demonstration of this fact.

4. On the existence of "zero-free" solutions. Formula (1) of Theorem 1 exhibits a relation between any pair of solutions of our phase problem. In this section we seek a single particularly simple solution; to wit, one whose holomorphic extension to the upper half-plane is free of zeros, so that $B(z) \equiv 1$ in (6). Such a "zero-free" solution always exists in $\mathcal{C}(a)$ whenever a solution exists in $\mathcal{C}(a)$ whose holomorphic extension has only a finite number of zeros, and we shall show that this is still true whenever some solution has a holomorphic extension with sufficiently sparse zeros, though possibly infinitely many.

The precise condition of sparseness of zeros in the upper half-plane which we take as an hypothesis is that the product $\prod_{\nu} \exp(2i \arg a_{\nu})$ be convergent, where $\{a_{\nu}\}$ is the set of all zeros of the holomorphic extension of some solution ϕ , i.e., $\phi \in \mathcal{C}(a)$, the limit being independent of the enumeration of the set $\{a_{\nu}\}$. Then the zeros must be on the whole rather close to the x -axis, and since we have proved that there cannot exist a finite cluster point of zeros we must have $|\operatorname{Re} a_{\nu}| \rightarrow +\infty$. Using this fact we conclude that

$$\prod_{\nu} \exp(i[\arg(a_{\nu} + i) + \arg(a_{\nu} - i)])$$

and

$$\prod_{\nu} \exp(2i \arg(x - a_{\nu})) = \prod_{\nu} \frac{x - a_{\nu}}{x - \bar{a}_{\nu}}$$

are convergent products, the latter being so uniformly with respect to finite x -intervals. It follows that the Blaschke product $B(z)$ associated with ϕ can be written in the form

$$B(z) = e^{i\eta} \cdot \prod_n \frac{z - a_n}{z - \bar{a}_n}, \quad y \geq 0, \eta \text{ real.}$$

A "zero-free" solution, q , if it exists in $\mathcal{C}(a)$ must, by (6), be of the form

$$\hat{q}(x) = e^{i\alpha + i\beta x} D(x), \quad D(x) = \lim_{y \rightarrow 0} D(x + iy).$$

We can afford to take $\alpha = \beta = 0$ as the factors $e^{i\alpha + i\beta x}$ are irrelevant for our present considerations, since they correspond to trivial transformations of q . Therefore, taking account of the above assumption on the distribution of the zeros $\{a_n\}$ of the holomorphic extension of $\hat{\phi}(x)$, we have, by Theorem 1, the necessary condition,

$$\widehat{\phi}(x) = \prod_n \frac{x - a_n}{x - \bar{a}_n} \cdot \widehat{q}(x), \quad \widehat{q}(x) = \prod_n \frac{x - \bar{a}_n}{x - a_n} \cdot \widehat{\phi}(x).$$

We proceed to show that such a function q does indeed exist in $\mathcal{C}(a)$. Put

$$\sigma_n(t) = \begin{cases} -2\gamma_n e^{-ia_n t} & \text{if } t \leq 0, \\ 0 & \text{if } t > 0, \end{cases}$$

where $a_n = x_n + iy_n$. Then

$$\frac{x - \bar{a}_n}{x - a_n} \cdot \widehat{\phi}(x) = \widehat{T_n \phi}(x),$$

where

$$T_n \phi = \sigma_n * \phi + \phi,$$

"*" denoting convolution in L^1 .

Regarding σ_n and ϕ as functions on $(-\infty, \infty)$, we have σ_n and ϕ belonging to L^1 . Hence $\sigma_n * \phi \in L^1$, $T_n \phi \in L^1$. Since $|\widehat{\phi}(x)| = |(T_n \phi)^\wedge(x)|$ and $\phi \in L^2$, it follows that $T_n \phi \in L^2$. By definition,

$$\begin{aligned} T_n \phi(t) &= \int_{-\infty}^{\infty} \phi(u) \sigma_n(t-u) du + \phi(t) \\ &= \int_{\max(0, t)}^{\infty} \phi(u) \sigma_n(t-u) du + \phi(t); \end{aligned}$$

and if $t < 0$, this is

$$T_n \phi(t) = -2\gamma_n e^{-ia_n t} \int_0^{\infty} e^{ia_n u} \phi(u) du = 0,$$

because a_n is a zero of the holomorphic extension of $\widehat{\phi}$. Obviously

$$\widehat{T_n \phi}(x) \neq 0$$

for all real x . Hence $T_n \phi \in \mathcal{C}(a)$. It follows at once that

$$\prod_{n=1}^N \frac{x - \bar{a}_n}{x - a_n} \cdot \widehat{\phi}(x)$$

is the Fourier transform $\widehat{\psi_N}(x)$ of the function

$$\psi_N(t) = T_1 T_2 \cdots T_N \phi(t),$$

and $\psi_N \in \mathcal{C}(a)$. Clearly, $\psi_N(x)$ converges pointwise everywhere as $N \rightarrow \infty$ to the function

$$\widetilde{\psi}(x) = \prod_n \frac{x - \bar{a}_n}{x - a_n} \cdot \widehat{\phi}(x),$$

and since $|\tilde{\psi}(x)| = |\hat{\phi}(x)|$, $\tilde{\psi}$ must belong to L^2 , and hence

$$\tilde{\psi} = \hat{q}, \quad q \in L^2.$$

Such q is in the first place defined only to within the class of functions differing from it on null sets. We shall show that under the present circumstances q can be chosen in $\mathcal{C}(a)$.

First, recall that $\hat{\psi}_N(x) \rightarrow \hat{q}(x)$ uniformly on finite x -intervals and that $|\hat{\psi}_N(x)| = |\hat{q}(x)|$ almost everywhere. Then, given $\epsilon > 0$, split as follows:

$$\int_{-\infty}^{\infty} |\psi_N(x) - q(x)|^2 dx = \int_{-\infty}^{-A} + \int_{-A}^A + \int_A^{\infty},$$

choosing A independent of N so that the first and third integrals on the right add up to less than ϵ . With such A fixed, for all N sufficiently large the second integral is less than ϵ . Hence $\|\hat{\psi}_N - \hat{q}\|_2 = \|\psi_N - q\|_2 \rightarrow 0$. Hence $q(t) = 0$ for almost all $t < 0$, and we can redefine q so that it is identically 0 for $t < 0$.

Clearly, $\hat{q}(x) \neq 0$, $-\infty < x < \infty$.

Since $\{\psi_N\}$ is a Cauchy sequence in L^2 , there exists a subsequence N_ν such that $\{\psi_{N_\nu}(t)\}$ is a Cauchy sequence of complex numbers for almost all t : $|\psi_{N_\nu}(t) - \psi_{N_\mu}(t)| \rightarrow 0$ as $\nu, \mu \rightarrow \infty$. Application of the diagonal process yields a further subsequence of $\{N_\nu\}$ (which we denote by the same notation) such that for a countable dense set of t 's

$$(7) \quad |\psi_{N_\nu}(t) - \psi_{N_\mu}(t)| \downarrow 0 \text{ as } \nu, \mu \rightarrow \infty.$$

Since the term $\phi(t)$ cancels out in $\psi_{N_\nu}(t) - \psi_{N_\mu}(t)$, this difference is a continuous function of t , and therefore (8) holds for all t . By the theorem of Lebesgue on integrating monotonic sequences, it follows that

$$\int_0^\infty |\psi_{N_\nu}(t) - \psi_{N_\mu}(t)| dt \rightarrow 0, \quad (\nu, \mu \rightarrow \infty).$$

Therefore ψ_{N_ν} tends to some function ψ_0 in L^1 . But ψ_{N_ν} tends to q in L^2 . Therefore $\psi_0(t) = q(t)$ almost everywhere, and hence $q \in L^1$. This completes the proof that $q \in \mathcal{C}(a)$. We can thus assert

THEOREM 2. *If some function $\phi \in \mathcal{C}(a)$ is such that the zeros $\{a_\nu\}$ of the holomorphic extension of $\hat{\phi}$ satisfy the condition:*

$$\prod_{\nu} e^{2i \arg a_\nu} \text{ is convergent to a limit which is independent of the enumeration of } \{a_\nu\},$$

then there exists a solution $q \in \mathcal{C}(a)$ whose Fourier transform is given by

$$\hat{q}(x) = e^{i\alpha + i\beta x} D(x), \quad -\infty < x < \infty,$$

where α, β are real and $D(x)$ is a continuous function given by

$$D(x) = \lim_{y \rightarrow 0} \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \frac{\log a(t)}{1 + t^2} dt \right\}.$$

5. **On the arbitrariness of $B(z)$.** We have seen that for functions ϕ in the class $\mathcal{C}(a)$ the canonical representation of the holomorphic extension of $\widehat{\phi}$ contains a certain Blaschke product, the limit function of which contributes to the phase of $\widehat{\phi}(x)$. It is natural to inquire whether the special properties of ϕ restrict the Blaschke products appearing, beyond the necessity of continuous boundary values. Explicitly, suppose $B(z)$ is any Blaschke product with continuous boundary values $B(x) = \lim_{y \rightarrow 0} B(x + iy)$.

Q_1 : Does there exist $\psi \in \mathcal{C}(a)$ such that

$$\widehat{\psi}(x) = B(x)a(x) \exp iH(x), \quad -\infty < x < \infty?$$

The function $H(x)$ is defined in the remark at the end of §3. A more specific question is

Q_2 . Given $\phi \in \mathcal{C}(a)$ and $B(z)$ as above, does there exist $\psi \in \mathcal{C}(a)$ such that $\psi(x) = B(x)\phi(x)$, $-\infty < x < \infty$?

We are unable to answer these questions. Their difficulty stems from the requirement that ψ belong to L^1 . That such ψ exists in L^2 and vanishes for negative arguments is almost trivial. We consider Q_1 . The function $B(x)a(x) \cdot \exp iH(x)$ is the boundary function of a function $\Phi(z)$ in the Hardy class H_2 ; that is Φ is holomorphic for $y > 0$,

$$\int_{-\infty}^{\infty} |\Phi(x + iy)|^2 dx \leq M < \infty, \quad M \text{ independent of } y,$$

$$\Phi(x) = \lim_{y \rightarrow 0+} \Phi(x + iy) = B(x)a(x) \exp iH(x),$$

almost everywhere. Therefore ([7, Theorem 93, p. 125]),

$$\begin{aligned} \Phi(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Phi(u)}{u - z} du, & (y > 0) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(u) \left[\frac{1}{-i(u - \bar{z})} \right] du \\ &= \int_0^{\infty} e^{izt} \psi(t) dt, \end{aligned}$$

where

$$\Phi(x) = \text{l.i.m.}_{A \rightarrow \infty} \int_{-A}^A e^{ixt} \psi(t) dt, \quad \psi \in L^2(-\infty, \infty).$$

On the other hand

$$(8) \quad \Phi(x) = \lim_{y \rightarrow 0} \Phi(x + iy),$$

and

$$\begin{aligned} & \left\{ \int_{-\infty}^{\infty} \left| \Phi(x) - \int_0^A e^{ixt} \psi(t) dt \right|^2 dx \right\}^{1/2} \\ & \leq \left\{ \int_{-\infty}^{\infty} \left| \Phi(x) - \int_0^{\infty} e^{ixt - y^t} \psi(t) dt \right|^2 dx \right\}^{1/2} \\ & \quad + \left\{ \int_{-\infty}^{\infty} \left| \int_0^{\infty} e^{ixt - y^t} \psi(t) dt - \int_0^A e^{ixt} \psi(t) dt \right|^2 dx \right\}^{1/2} = J_1 + J_2. \end{aligned}$$

By (8), $J_1 = o(1)$, $y \rightarrow 0$. By Plancherel, for $A > 0$,

$$\int_{-\infty}^{\infty} \left| \int_A^{\infty} e^{ixt - y^t} \psi(t) dt \right|^2 dx = 2\pi \int_A^{\infty} |\psi(t)|^2 e^{-2y^t} dt.$$

Hence, if $\epsilon > 0$, we can find A_0 such that $A > A_0$ implies that the last integral is $< \epsilon$. Fixing such A ,

$$\begin{aligned} J_2 & \leq \left\{ \int_{-\infty}^{\infty} \left| \int_0^A e^{ixt} (1 - e^{-y^t}) \psi(t) dt \right|^2 dx \right\}^{1/2} + \epsilon^{1/2} \\ & = \left\{ \int_0^A (1 - e^{-y^t})^2 |\psi(t)|^2 dt \right\}^{1/2} + \epsilon^{1/2} = o(1) + \epsilon^{1/2}, \quad y \rightarrow 0. \end{aligned}$$

Therefore

$$\Phi(x) = \lim_{A \rightarrow \infty} \int_0^A e^{ixt} \psi(t) dt,$$

and by the uniqueness of Fourier transforms we conclude that $\psi(t) = 0$ for $t < 0$ (after possibly altering ψ on a null set)⁽⁴⁾. The same procedure is valid for Q_2 .

6. Examples. Finally we shall exhibit a couple of examples which serve to illustrate Theorem 1.

I. The first example is a generalization of one which was pointed out to me by Professor W. Rudin before Theorem 1 was found. Suppose that $\phi \in \mathcal{C}(a)$, and furthermore $\phi(t) = 0$ for $t < n$ where n is a positive integer. Let $a_1, a_2, \dots, a_n$ be arbitrary real numbers, and let λ be any nonreal complex number. Then the function

$$\phi_{\lambda}(t) = \lambda \phi(t) + \sum_{k=1}^n a_k (\phi(t + k) + \phi(t - k))$$

⁽⁴⁾ This result can also be derived by use of (D), (iv), in conjunction with a theorem of Paley and Wiener [2, Theorem XII].

belongs to $\mathfrak{C}(a)$, and its Fourier transform is

$$F_\lambda(x) = \left(\lambda + 2 \sum_{k=1}^n a_k \cos kx \right) \hat{\phi}(x).$$

Hence

$$(9) \quad F_\lambda(x) = \frac{P(e^{ix}) - \lambda}{P(e^{ix}) - \bar{\lambda}} F_{\bar{\lambda}}(x),$$

where we have put

$$P(e^{iz}) = -2 \sum_{k=1}^n a_k \cos kx.$$

Replacing x by $z = x + iy$, $y > 0$, in (9) we obtain the corresponding relation for the holomorphic extensions of F_λ and $F_{\bar{\lambda}}$. Since $\text{Im } \lambda \neq 0$, $P(w) - \lambda$ vanishes for $2n$ nonreal values of w , none of which can be of modulus 1, and which must therefore be of the form $w_1, w_2, \dots, w_n, 1/w_1, 1/w_2, \dots, 1/w_n$, where $|w_1| < 1, \dots, |w_n| < 1$. Therefore the only values of z in the upper half-plane for which $P(e^{iz}) - \lambda$ vanishes are

$$z = a_n^{(k)} \equiv -i \log |w_k| + \arg w_k + 2\pi n, \quad n = 0, \pm 1, \dots, k = 1, 2, \dots, n.$$

Hence the Blaschke product $B_1(z)$ formed with these zeros is convergent. Likewise $P(e^{iz}) - \bar{\lambda} = 0$, $y > 0$, if and only if

$$z = b_n^{(k)} \equiv -i \log |w_k| - \arg w_k + 2\pi n, \quad n = 0, \pm 1, \dots, k = 1, 2, \dots, n,$$

so that the corresponding Blaschke product $B_2(z)$ is defined. Put, for $y > 0$,

$$Q(z) = \frac{B_2(z)}{B_1(z)} \cdot \frac{P(e^{iz}) - \lambda}{P(e^{iz}) - \bar{\lambda}}.$$

Then $Q(z)$ is holomorphic and fails to vanish for $y > 0$. We shall show that $Q(z)$ is bounded in the strip $|x| \leq 2\pi$, $y > 0$. By periodicity it will then follow that $Q(z)$ is bounded in the entire upper half-plane. (Note that $B_1(z)$, $B_2(z)$ as well as $P(e^{iz})$ are periodic with period 2π .) Put

$$K = 1 + \max_{k=1, \dots, n} \{ -\log |w_k| \}.$$

Then for $y \geq K$, $|x| \leq 2\pi$, $(P(e^{iz}) - \lambda)/(P(e^{iz}) - \bar{\lambda})$ is bounded, and in the rectangle

$$|x| \leq 2\pi, \quad 0 < y \leq K$$

$Q(z)$ is bounded. It remains to prove that $B_2(z)/B_1(z)$ is bounded for $z \in S_K$: $y \geq K$, $|x| \leq 2\pi$, and it is no real loss in generality to take $k=1$, so that the

zeros $a_n^{(k)} \equiv a_n$, $b_n^{(k)} \equiv b_n$ lie on a single horizontal line in the upper half-plane. Thus

$$\left| \frac{B_2(z)}{B_1(z)} \right| = \prod_n \left| \frac{z - b_n}{z - \bar{b}_n} \cdot \frac{z - \bar{a}_n}{z - a_n} \right|.$$

Let us pair the adjacent zeros lying to the left of S_K into pairs (b_n, a_n) where b_n lies nearer to S_K than the adjacent a_n , and let $\prod_1$ denote the partial product containing the zeros so paired. For $z \in S_K$,

$$\left| \frac{z - b_n}{z - a_n} \right| \leq \left| \frac{z - \bar{b}_n}{z - \bar{a}_n} \right|,$$

or

$$\left| \frac{z - b_n}{z - \bar{b}_n} \cdot \frac{z - \bar{a}_n}{z - a_n} \right| \leq 1.$$

Hence $\prod_1 \leq 1$. We pair the zeros lying to the right of S_K in a similar way and form $\prod_2$, $\prod_2 \leq 1$. This leaves out of account approximately seven zeros which lie nearest the y -axis, but the partial product $\prod_3$ involving these is clearly bounded for $z \in S_K$. Hence

$$\left| \frac{B_2(z)}{B_1(z)} \right| = \prod_1 \prod_2 \prod_3$$

is bounded for $z \in S_K$.

Therefore $|Q(z)| < M_0$ for $y > 0$. Putting $M(r) = \text{Max}_{|z|=r} |Q(z)|$,

$$\liminf_{r \rightarrow \infty} \frac{\log M(r)}{r} \leq 0.$$

But

$$\limsup_{y \rightarrow 0} |Q(z)| \leq 1, \quad -\infty < x < \infty.$$

Hence it follows by the Phragmén-Lindelöf theorem (E) that

$$(10) \quad |Q(z)| \leq 1 \text{ for } y > 0.$$

Next we show that

$$(11) \quad \int_{-\infty}^{\infty} \frac{\log |Q(z)|}{1+x^2} dx \rightarrow 0, \quad y \rightarrow 0.$$

According to (A),

$$\int_{-\infty}^{\infty} \frac{\log |B_j(x)|}{1+x^2} dx \rightarrow 0, \quad y \rightarrow 0, \quad j = 1, 2.$$

By the periodicity of $(P(e^{iz}) - \lambda)/(P(e^{iz}) - \bar{\lambda})$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left| \log \frac{P(e^{iz}) - \lambda}{P(e^{iz}) - \bar{\lambda}} \right| \frac{dx}{1+x^2} &= \sum_{k=-\infty}^{\infty} \int_{2\pi k}^{2\pi(k+1)} \left| \log \frac{P(e^{iz}) - \lambda}{P(e^{iz}) - \bar{\lambda}} \right| \frac{dx}{1+x^2} \\ &= \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{1+(x+2k\pi)^2} \cdot \left| \log \frac{P(e^{iz}) - \lambda}{P(e^{iz}) - \bar{\lambda}} \right| dx, \end{aligned}$$

which clearly tends to 0 as $y \rightarrow 0$. By (10) and (11), since Q has no zeros, (A) implies that $Q(z)$ must be of the form

$$Q(z) = e^{ic+i\beta z}, \quad c \text{ real, } \beta \geq 0,$$

so that

$$\frac{P(e^{iz}) - \lambda}{P(e^{iz}) - \bar{\lambda}} = e^{ic+i\beta z} \cdot \frac{B_1(z)}{B_2(z)}.$$

Passing to the limit $y \rightarrow 0$ this shows that (9) can be written

$$B_2(x)F_{\lambda}(x) = e^{ic+i\beta x}B_1(x)F_{\bar{\lambda}}(x),$$

as required by Theorem 1.

II. In a rather well-known problem arising in crystallography the function ϕ would stand for electron density, and would be therefore non-negative. We shall show by a simple example that even if we assume $\phi \geq 0$, in addition to $\phi \in \mathcal{C}(a)$, ϕ is *not* uniquely determined.

Let $b = \alpha + i\beta$ be a complex number such that $\beta > 0$, $\alpha \neq 0$, $|4\beta/\alpha| < 1$. Put $\phi_1(t) = e^{-\beta t}$ for $t \geq 0$, $\phi_1(t) = 0$ for $t < 0$, and put $a(x) = |1/(ix - \beta)|$. Define $\phi_2(t)$ by the condition

$$\hat{\phi}_2(x) = \frac{x+b}{x+b} \cdot \frac{x-b}{x-b} \cdot \hat{\phi}_1(x), \quad -\infty < x < \infty.$$

Then $\phi_1, \phi_2 \in \mathcal{C}(a)$, and a calculation shows that

$$\begin{aligned} \phi_2(t) &= \phi_1(t) - 4\beta \int_0^t \cos \alpha(t-u) e^{-\beta(t-u)} \phi_1(u) du \\ &\quad + \frac{4\beta^2}{\alpha} \int_0^t \sin \alpha(t-u) e^{-\beta(t-u)} \phi_1(u) du \end{aligned}$$

if $t \geq 0$. If we substitute the particular ϕ_1 defined above, it turns out that

$$\phi_2(t) = e^{-\beta t} \left(\left(1 - \frac{4\beta}{\alpha} \sin \alpha t \right) + \frac{4\beta^2}{\alpha^2} (1 - \cos \alpha t) \right) \text{ if } t \geq 0.$$

Thus $\phi_2(t) \geq 0$.

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